Probability Prelim Exam for Actuarial Students
January 2024

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for mathematically incorrect statements.

1. (10 points) Three persons A, B, and C take turns to flip a fair coin. Person A flips the coin first, then B, then C, then A, and so on so forth. The first person to get a head wins. Determine the corresponding probability space and calculate the probability of the event $\{A \text{ wins}\}$.

2. (10 points) Consider the probability space $([0,1], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra of subsets of $[0,1]$ and λ is the Lebesgue measure on $[0,1]$. Let X be a random variable on the probability space that is defined as
$$X(\omega) = I_{[0,0.2]}(\omega) - 2I_{[0.4,0.8]}(\omega),$$
where I is the indicator function. Calculate $\operatorname{Var}(X)$.

3. (10 points) Let X be a nonnegative and integrable random variable. Show that
$$\lim_{x\to\infty} xP(X > x) = 0.$$

4. (10 points) Given a sequence of random variables $\{X_n\}_{n\geq 1}$ and a random variable X defined on the same probability space such that $X_n \geq 0$, $E[X_n] = 1$, $X_n \to X$ a.s., and $\lim_{n\to\infty} E[X_n] \neq E[X]$.

5. (10 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with $E[|X_1|] < \infty$. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences of positive integers both tending to infinity. Let
$$M_n = \frac{1}{b_n} \sum_{k=a_n}^{a_n+b_n-1} X_k, \quad n \geq 1.$$
Show that if $\{a_n/b_n\}_{n\geq 1}$ is bounded, then $\lim_{n\to\infty} M_n = E[X_1]$ a.s.

6. (10 points) Let X and Y be independent random variables with the following distribution:
$$P(X = 1) = P(Y = 0) = \frac{1}{3}, \quad P(X = 0) = P(Y = 1) = \frac{2}{3}.$$
Calculate $E[X + Y|X - Y]$.

7. (10 points) Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Let $t < s < u$. Compute
$$E\left[B_t B_s^2 B_u\right].$$