Abstract Algebra Prelim Aug. 2024

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) Let $m, n \geq 2$ in $\mathbf{Z}$ and let $u \in (\mathbf{Z}/(m))^\times$ satisfy $u^n \equiv 1 \bmod m$, so we get a group homomorphism $\varphi : \mathbf{Z}/(n) \to (\mathbf{Z}/(m))^\times$ by $\varphi(x \bmod n) = u^x \bmod m$. Let $G = \mathbf{Z}/(m) \rtimes_\varphi \mathbf{Z}/(n)$ be the semidirect product associated to φ.

   (a) (**4 pts**) Fill in the missing part of the group law **and** inversion in G: $(a,b)(c,d) = (?, b+d)$ and $(a,b)^{-1} = (?, -b)$.

   (b) (**6 pts**) Show the center of G is $\{(a,b) \in G : (u-1)a \equiv 0 \bmod m \text{ and } u^b \equiv 1 \bmod m.\}$.

2. (**10 pts**) Let R be a nonzero commutative ring, $G = \mathrm{GL}_2(R) = \{A \in \mathrm{M}_2(R) : \det A \in R^\times\}$, and $H = \{(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}) : x \in R^\times, y \in R\}$. You may use without proof that H is a subgroup of G.

   Show the normalizer of H in G is the set U of upper-triangular matrices in G:
$$\mathrm{N}_G(H) = U = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in R^\times, d \in R^\times, b \in R \right\}.$$

3. (**10 pts**) Let $D \in \mathbf{Z}$ be odd and not a square. Let I be the ideal $(2, 1+\sqrt{D})$ in $\mathbf{Z}[\sqrt{D}]$.

   (a) (**6 pts**) Show $I^2 = 2I$ if $D \equiv 1 \bmod 4$ and $I^2 = (2)$ if $D \equiv 3 \bmod 4$.

   (b) (**4 pts**) If $D \equiv 1 \bmod 4$, then show I is a non-principal ideal. Part (a) can be used here.

4. (**10 pts**)

   (a) (**5 pts**) (Division by monics) Let R be a nonzero commutative ring. For $f(x)$ and $g(x)$ in $R[x]$ such that $g(x)$ is monic, prove that there are $q(x)$ and $r(x)$ in $R[x]$ such that (**i**) $f(x) = g(x)q(x) + r(x)$ and (**ii**) $r(x) = 0$ or $\deg r < \deg g$.
   Note: You are not being asked to show $q(x)$ and $r(x)$ are unique, although they are.

   (b) (**5 pts**) For $D, N \in \mathbf{Z}$ with D not a square and $N \geq 2$, let $\varphi : \mathbf{Z}[x] \to \mathbf{Z}[\sqrt{D}]/(N)$ by $\varphi(f(x)) = f(\sqrt{D}) \bmod (N)$. You may use without proof that φ is a ring homomorphism. Prove φ is surjective and $\ker \varphi$ is the ideal $(N, x^2 - D)$. Part (a) can be helpful.

5. (**10 pts**) For finite-dimensional vector spaces V and W over a field k, let $L : V \to W$ be a k-linear map and $L^* : W^* \to V^*$ be its dual map.

   (a) (**4 pts**) Prove that if L is surjective then L^* is injective.

   (b) (**6 pts**) Prove that if L is injective then L^* is surjective. (Hint: Extend a basis of $L(V)$ to a basis of W and let U be the span of the extra basis vectors, so $W = L(V) \oplus U$.)

6. (**10 pts**) Give examples as requested, with justification.

   (a) (**2.5 pts**) An integer $m \geq 2$ such that the additive group $\mathbf{Z}/(m)$ has an automorphism of order 3.

   (b) (**2.5 pts**) A polynomial $f(x) = x^2 + bx + c$ in $\mathbf{Z}[x]$ such that the ideal $(5, f(x))$ in $\mathbf{Z}[x]$ is maximal.

   (c) (**2.5 pts**) Gaussian integers γ and ρ such that $10+7i = (2+5i)\gamma + \rho$ and $\mathrm{N}(\rho) < \mathrm{N}(2+5i)$.

   (d) (**2.5 pts**) An orthonormal basis of the two-dimensional vector space $V = \mathbf{R} + \mathbf{R}x$ with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$.