Applied Math Prelim August 2024

Student ID

1. Let X be a normed linear space.
   (a) (4 pts) State the definition of a sequence $\{x_n\} \subset X$ converges weakly to an element $x \in X$.
   (b) (8 pts) Show that a weakly convergent sequence is bounded.
   (c) (8 pts) If $x_n \rightharpoonup x$, then a sequence of linear combinations of x_n converges strongly to x.
2. Let $\mathcal{H}$ be a Hilbert space and $\{\phi_n : n \in N\}$ be an orthonormal system of $\mathcal{H}$. Let T be a bounded linear operator on $\mathcal{H}$.
   (a) (5 pts) Show that $T\phi_n \rightharpoonup 0$ weakly.
   (b) (5 pts) If T is compact, show that $\lim_{n\to\infty} ||T\phi_n|| = 0$.
   (c) (10 pts) Let λ_n be a sequence of complex numbers. Show that the operator S defined by $Sf = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n$ is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$.
3. (a) (4 pts) State the definition of Fréchet derivative.
   (b) (6 pts) Define $f : C[0,1] \to C[0,1]$ by $(f(x))(t) = \int_0^1 g(t, x(s))ds$, where g is a function of two variables whose second partial derivatives with respect to the second argument is continuous. Compute the Fréchet derivative of f.
   (c) (10 pts) If the bounded linear map A satisfies the weaker condition

$$\lim_{\lambda\to 0} \frac{1}{\lambda} ||f(x + \lambda h) - f(x) - \lambda Ah|| = 0$$

   for every $h \in X$, then f is said to be Gâteaux differentiable at x with A being the Gâteaux derivative at x. Prove that if f is Fréchet differentiable at x, then it is Gâteaux differentiable at x and the two derivatives are equal.
4. Let f be a locally integrable function on R.
   (a) (5 pts) Interpret f as a distribution on R.
   (b) (5 pts) Define δ as a distribution.
   (c) (10 pts) Find a distribution T on R such that $\partial^2 T + T = \delta$.
5. (20 pts) Given that a closed and bounded set K in l^2 is compact if and only if

$$\lim_{n\to\infty} \sup_{x\in K} \sum_{i\geq n} |x(i)|^2 = 0.$$

   Prove that $K = \{x \in l^2, |x(i)| \leq \frac{1}{i}\}$ is compact.