COMPLEX ANALYSIS PRELIM, AUGUST 2024

Instructions

- To *prove or disprove a statement* means either to prove the statement or to provide a counterexample for it.
- The terms "holomorphic" and "analytic" are used interchangeably.
- The set of complex numbers is denoted by $\mathbb{C}$. Denote $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$. For two sets A, B, denote $A \setminus B = \{x \in A; x \notin B\}$.

1. How many zeros counting multiplicities does the function $g(z) = 2z^7 - 9e^{iz} + 1$ have in the region $\{z \in \mathbb{C} : 1 < |z| < 2\}$? Prove your assertion.

2. Explicitly construct a one-to-one conformal map from $\mathbb{D} \setminus [-1, 0]$ onto $\mathbb{D}$. Here $[-1, 0]$ denotes the set $\{z \in \mathbb{C} : -1 \leq \mathrm{Re}(z) \leq 0,\ \mathrm{Im}(z) = 0\}$.

3. Let $B_R = \{z \in \mathbb{C} : |z| < R\}$ for some $R > 2$. Assume polynomial $p(z) = a_{100}z^{100} + \cdots + a_1 z + a_0$ satisfies $\sup_{B_R} |p| \leq 1$. Show that
$$|a_k| \leq \frac{1}{R^k} \quad \text{for each } 0 \leq k \leq 100.$$

4. Prove or disprove the statement: Let $G = \mathbb{D} \setminus \{0, i/2\}$. Then, the holomorphic automorphism group $\mathrm{Aut}(G)$ consists of the identity map only; that is, if a holomorphic map $\varphi : G \to G$ is one-to-one and onto, then $\varphi(z) = z$ for all $z \in G$.

5. Let f be a holomorphic function on $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > -1\}$. Assume that
$$|f(0)| \geq |f(z)| \quad \text{for any } z \text{ with } |z| < 1,$$
and that
$$|f(0)| \leq \frac{1}{3}|f(2024i)|.$$
Is f necessarily identically equal to zero on H? Prove your assertion.

6. Find the limit
$$\lim_{R \to +\infty} \left(\int_0^\pi e^{R \sin\theta + iR\cos\theta} d\theta \right)$$
and justify your answer, or show that the limit does not exist.

7. For any holomorphic function $h : \mathbb{D} \to \mathbb{D}$, show that
$$|h''(0)| \leq 2\left[1 - |h(0)|^2\right].$$