Real Analysis Preliminary Exam
August 2024

Instructions and Notation

- Justify all your steps, clearly identifying all results you are using. Do not invoke a result that is essentially equivalent to what you are asked to prove.
- The Lebesgue measure on $\mathbb{R}$ is represented by the letter m.

Problems

1. (10 points) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \to [-\infty, \infty]$ be a measurable function satisfying $\int e^f d\mu < \infty$. Prove
$$\lim_{x\to\infty} e^x \mu(\{f \geq x\}) = 0.$$

2. (10 points) Let $(\mu_m : m \in \mathbb{N})$ be measures on the sigma-algebra $\mathcal{F}$. Prove that $\sum_{m=1}^{\infty} \mu_m$ is a measure on $\mathcal{F}$.

3. (10 points) Calculate the following limit of Riemann integrals:
$$\lim_{n\to\infty} \int_0^2 \left(\frac{x}{1+x^{3n}}\right)^{1/n} dx.$$

4. (10 points) Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous with $\int_{[0,1]} |f'|^p dm < \infty$ for some $p \in (1, \infty)$. Prove that there exists a nonnegative function h on $(0,1]$ satisfying $\lim_{t\to 0} h(t) = 0$ such that
$$\sup_{0\leq x<y\leq 1} \frac{|f(x)-f(y)|}{(y-x)^{1-\frac{1}{p}} h(y-x)} \leq 1.$$

5. (10 points) Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to [0, \infty]$ be $\mathcal{F} \times \mathcal{G}$-measurable and let $g : Y \to [0, \infty]$ be $\mathcal{G}$-measurable.

   (a). Prove
$$\int_Y \left(\int_X f(x,y) d\mu(x)\right) g(y) d\nu(y) \leq \int_X \sqrt{\int_Y f^2(x,y) d\nu(y)} d\mu(x) \times \sqrt{\int_Y g^2(y) d\nu(y)}. \tag{2}$$

   (b). Use (2) to obtain
$$\sqrt{\int_Y (\int_X f(x,y) d\mu(x))^2 d\nu(y)} \leq \int_X \sqrt{\int_Y f^2(x,y) d\nu(y)} d\mu(x).$$

6. (10 points) Let μ and ν be finite measures on the measurable space $(\Omega, \mathcal{F})$.

   (a). State the Lebesgue Decomposition Theorem for ν with respect to μ.

   (b). Let $g = \frac{d\nu}{d(\nu+\mu)}$. Prove that the Lebesgue decomposition of ν with respect to μ is given by:
$$d\nu = \frac{g}{1-g} d\mu + \mathbf{1}_{\{g=1\}} d\nu.$$

7. (10 points) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable bounded function satisfying $\varphi(x+1) = \varphi(x)$ for all $x \in \mathbb{R}$ and $\int_{[0,1]} \varphi dm = 0$. Prove that for $f \in L^1(m)$,
$$\lim_{n\to\infty} \int f(x)\varphi(nx) dm(x) = 0.$$