GEOMETRY AND TOPOLOGY PRELIM, AUGUST 2024

1. Prove that if a topological space is Lindelöf (i.e., every open cover has a countable subcover), then any closed subset of it is Lindelöf as well.

2. Let $f : X \to Y$ be an open and surjective map between topological spaces X and Y. Assume that Y is connected and that for each $y \in Y$, the set $f^{-1}(y) \subset X$ is connected. Prove that X is also connected.

3. Let E be a closed subset of a compact Hausdorff space X and let $Y = X \setminus E$ have the subspace topology. Prove that the one-point compactification of Y is homeomorphic to the quotient space X/E.

4. Compute the fundamental group of the space X obtained from the disjoint union of two tori $\mathbb{T}^2$ by identifying them along a pair of points, i.e.,

$$X = \mathbb{T}^2 \sqcup \mathbb{T}^2 /_{x_1 \sim y_1,\, x_2 \sim y_2}$$

for x_1, x_2 on one copy of $\mathbb{T}^2$ and y_1, y_2 on the other.

5. Prove that any continuous map $f : \mathbb{RP}^2 \to \mathbb{S}^1$ is nullhomotopic.

6. Let $p : X \to Y$ be a covering map, where X is path connected and Y is simply connected. Prove that p is a homeomorphism.