Abstract Algebra Prelim Jan. 17, 2025

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. **(10 pts)** Let a group G act on the set X. For $x \in X$, let $\mathrm{Stab}_x = \{g \in G : gx = x\}$ denote the stabilizer subgroup of x.

   (a) **(5 pts)** If x and y in X are in the same G-orbit, then show Stab_x and Stab_y are conjugate subgroups in G.

   (b) **(5 pts)** For $x \in X$, prove there is a bijection between the orbit of x and G/Stab_x, the left cosets of Stab_x in G.
   Part (b) does not depend on part (a).

2. **(10 pts)** Let G be a group and $g \in G$ have finite order n.

   (a) **(5 pts)** For $m \in \mathbf{Z}^+$, prove $g^m = e$ if and only if $n \mid m$.

   (b) **(5 pts)** For $k \in \mathbf{Z}^+$, prove g^k has order $n/(n, k)$.

3. **(10 pts)** Let $d \in \mathbf{Z}$ not be a square and $\mathrm{N}\colon \mathbf{Z}[\sqrt{d}] \to \mathbf{Z}$ be the norm: $\mathrm{N}(x + y\sqrt{d}) = x^2 - dy^2$ when $x, y \in \mathbf{Z}$. You may use without proof that $\mathrm{N}(\alpha\beta) = \mathrm{N}(\alpha)\,\mathrm{N}(\beta)$ for all α and β in $\mathbf{Z}[\sqrt{d}]$.

   (a) **(5 pts)** Show u in $\mathbf{Z}[\sqrt{d}]$ is a unit if and only if $\mathrm{N}(u) = \pm 1$.

   (b) **(5 pts)** If α in $\mathbf{Z}[\sqrt{d}]$ satisfies $\mathrm{N}(\alpha) = \pm p$ where p is a prime number, then prove α is irreducible in $\mathbf{Z}[\sqrt{d}]$.

4. **(10 pts)** Let F be a field and V be an n-dimensional F-vector space, with dual space V^*. Fix n vectors $v_1, \ldots, v_n$ in V and let $f\colon V^* \to F^n$ be the F-linear map where

$$f(\varphi) = \begin{pmatrix} \varphi(v_1) \\ \vdots \\ \varphi(v_n) \end{pmatrix} \quad \text{for each } \varphi \in V^*.$$

   (You may use the F-linearity of f without proof.)
   Show $\{v_1, \ldots, v_n\}$ is an F-linearly independent set if and only if f is injective.

5. **(10 pts)** Let A be an $n \times n$ real matrix that is diagonalizable. (Being diagonalizable means there is some basis of $\mathbf{R}^n$ such that the matrix representation of A in that basis is diagonal, *not* that A is a diagonal matrix in the standard basis.)

   (a) **(5 pts)** Prove that if $A = B^2$ for some diagonalizable real $n \times n$ matrix B, then every eigenvalue of A is nonnegative.

   (b) **(5 pts)** Prove a converse to part (a): if every eigenvalue of A is nonnegative, then $A = B^2$ for some diagonalizable real $n \times n$ matrix B. This does not depend on part (a).

6. **(10 pts)** Give examples as requested, with justification.

   (a) **(2.5 pts)** A group homomorphism $\mathbf{Z}/12\mathbf{Z} \to \mathbf{Z}/18\mathbf{Z}$ that is not identically zero.

   (b) **(2.5 pts)** A 2-Sylow subgroup of the dihedral group of order 12.

   (c) **(2.5 pts)** A maximal ideal in $\mathbf{Z}[x]$ that contains $x^2 + 1$.

   (d) **(2.5 pts)** A ring homomorphism $f\colon \mathbf{Z}[\sqrt{2}] \to \mathbf{Z}/7\mathbf{Z}$. (Describe f by its values $f(a + b\sqrt{2})$ for $a, b \in \mathbf{Z}$ and make sure that $f(1) \equiv 1 \bmod 7$.)