COMPLEX ANALYSIS PRELIM, JANUARY 2025

Instructions

- The terms "holomorphic" and "analytic" are used interchangeably.
- The set of complex numbers is denoted by $\mathbb{C}$. Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For two sets A, B, denote $A \setminus B = \{x \in A : x \notin B\}$.

1. How many zeros counting multiplicities does the function $e^z - z^2 + 2025i$ have on the left half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$? Prove your assertion.

2. Explicitly construct a one-to-one conformal map from the region $\{z \in \mathbb{C} : |z+i| > 1, |z+2i| < 2\}$ onto $\mathbb{D}$.

3. For any holomorphic function h on $\mathbb{D}$ such that $\mathrm{Im}[h(z)] \geq 3$ for all $z \in \mathbb{D}$ and $h(0) = 4i$, find the largest possible value of $|h'(0)|$.

4. Let $\mathscr{F}$ be the family of holomorphic functions defined on $\mathbb{D}$ satisfying
$$\int_{\{|x+iy|<1\}} |f(x+iy)|dxdy \leq 100, \quad \text{for all } f \in \mathscr{F}.$$
Show that $\mathscr{F}$ is a normal family.

5. Evaluate the integral
$$\int_0^{+\infty} \frac{x^{\frac{3}{\pi}}}{x^2+1} dx$$
and justify your answer.

6. Does there exist a harmonic function $u : \mathbb{C} \to \mathbb{R}$ such that there are constants $A, B > 0$ with
$$u(z) \leq A \log|z| + B, \quad \text{for all } z \in \left\{z \in \mathbb{C} : |z| > 1, |\mathrm{Im}(z)| \leq 10^{1000}\right\},$$
and that $\partial u/\partial x \neq 0$ and $\partial u/\partial y \neq 0$ at some $z_0 \in \mathbb{C}$? Prove your assertion.

7. Let g be a holomorphic function on $\mathbb{D} \setminus \{0\}$, and denote $g_k(z) = g(z/2^k)$ for each positive integer k. Suppose that
$$\max_{|z|=1/2} |g_k(z)| \leq k \quad \text{for all } k \geq 1.$$
Show that g can be extended to a holomorphic function on $\mathbb{D}$.