GEOMETRY & TOPOLOGY PRELIMINARY EXAM, JANUARY 2025

Problem 1. (a) Let X and Y be two topological spaces. Prove that a function $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for any set $A \subset X$. Here $\overline{A}$ is the closure of the set A.

(b) Prove that a continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.

Problem 2. Let $n \geq 2$ and $\mathbb{S}^n$ be an n-dimensional sphere in $\mathbb{R}^{n+1}$. Suppose that $f : \mathbb{S}^n \to \mathbb{S}^n$ is a continuous map such that $f(x) \neq -x$ for any $x \in \mathbb{S}^n$. Prove that the map f must be homotopic to the identity map on $\mathbb{S}^n$.

Problem 3. (a) Let $X \subset \mathbb{R}^3$ be the space defined by

$$X = \left\{(x, y, z) : x^2 + y^2 + z^2 = 1\right\} \cup \left\{(x, y, z) : x^2 + y^2 \leq 1,\ z = 0\right\}$$

and $A = \{(x, y, z) : x^2 + y^2 = 1,\ z = 0\}$. Prove that there exists no retraction from X to A.

(b) Let X be the subspace of $\mathbb{R}^3$ equal to the union of the unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with two line segments

$$\{(0, 0, z) : |z| \leq 1\} \cup \{(0, y, 0) : |y| \leq 1\}.$$

Compute the fundamental group of X. Show all work.

Problem 4. Let X, Y be connected topological spaces and $A \subset X$ and $B \subset Y$ be proper subsets (a proper subset means non-empty set and not equal to the whole space). Prove that $(X \times Y) \setminus (A \times B)$ is also connected.

Problem 5. Prove or disapprove the following statements. Show rigorous work.

(a) $\mathbb{R}^4$ is not homeomorphic to $\mathbb{R}^2$.

(b) There is a continuous map $f : \mathbb{S}^4 \to \mathbb{S}^1$ that is not null-homotopic.

Problem 6. Let f and g be continuous maps from a topological space X to a Hausdorff space Y. Prove that the set of points x for which $f(x) = g(x)$ is a closed subset of X.