Loss Models Prelims for Actuarial Students
Monday, January 13, 2025, 9:00 AM - 1:00 PM

Instructions:

1. There are five (5) equally-weighted questions and you are to answer all five.
2. Hand-held calculators are permitted.
3. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
4. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

Question No. 1:

Let X and Y denote two (non-negative) random losses, and assume that their survival functions are given, respectively, by

$$S_X(x) = \begin{cases} 0.3, & \text{for all } 0 \leq x < 2, \\ 0.1, & \text{for all } 2 \leq x < 4, \\ 0, & \text{for all } x > 4, \end{cases}$$

and

$$S_Y(y) = \left(\frac{2}{2+y} \right)^3, \quad \text{for all } y \geq 0.$$

(a) Calculate the mean and variance of X.

(b) Calculate the mean and variance of Y.

(c) Calculate VaR and CVaR at 80% level for both X and Y.

Question No. 2:

The probability generating function (pgf) of a random variable X is defined to be

$$P_X(z) = \mathbb{E}[z^X].$$

(a) Prove, by recursion, that the pgf generates the probabilities for a discrete random variable N, i.e., prove that

$$p_n = \Pr(N = n) = \left(\frac{1}{n!} \right) P_N^{(n)}(0) = \left(\frac{1}{n!} \right) \frac{d^n}{dz^n} P_N(z) \bigg|_{z=0},$$

for $n = 0, 1, 2, \ldots$

(b) Use the result in (a) to describe the distribution of N with pgf

$$P_N(z) = \frac{z}{5}(2 + 3z^2).$$

(c) Consider the aggregate loss models defined by

$$S_N = X_1 + X_2 + \cdots + X_N,$$

where $X_1, X_2, \ldots$ are identically, independent random variables describing claim severities, and N is a discrete random variable describing claim frequencies. All random variables are independent. Prove that the pgf of S_N can be expressed as

$$P_{S_N}(z) = P_N(P_X(z)).$$

Question No. 3:

Consider a collective risk model

$$S = \sum_{i=1}^{N} X_i,$$

in which N follows a Poisson distribution with mean 10. Regarding the loss X, it is known that

$\mathbb{E}[X] = 40,$
$\mathbb{E}[X \wedge 20] = 15,$
$\Pr(X > 20) = 0.8,$ and
$\mathbb{E}[X^2|X > 20] = 4,000.$

Now, assume that a deductible of $d = 20$ (per loss) is applied to all policies, resulting in a new collective risk model for the aggregate payment

$$\widetilde{S} = \sum_{i=1}^{\widetilde{N}} \widetilde{X}_i,$$

in which $\widetilde{X}_i := (X_i - d)|X_i > d$, and $\widetilde{N}$ counts the number of *payment* events.

(a) Show that $\widetilde{N}$ follows a Poisson distribution and compute its mean.

(b) Calculate the mean and variance of $\widetilde{S}$.

Question No. 4:

Individual ground-up loss amount X follows a two-parameter Weibull distribution with $\tau = 1/3$ and mean 60. See Appendix for parameterization of Weibull in terms of τ and θ. An insurance policy on X has a deductible amount of 5 and a policy limit of 120 per loss.

Assume loss amount increased due to inflation by 5% uniformly.

(a) Determine the value of the parameter θ.

(b) Calculate the expected value of claims per loss before the inflation. You may leave your answer in terms of the incomplete gamma function.

(c) Calculate the expected value of claims per loss after the inflation. You may leave your answer in terms of the incomplete gamma function.

(d) Calculate the variance of claims per loss after the inflation. You may leave your answer in terms of the incomplete gamma function.

Question No. 5:

Claim frequency N is being modeled as a zero-inflated Poisson expressed as follows:

$$\begin{aligned} p_0^m &= \pi + (1-\pi)p_0 \\ p_k^m &= (1-\pi)p_k, \quad \text{for } k = 1, 2, \ldots \end{aligned}$$

where $p_0 = e^{-\lambda}$ and $p_k = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k = 1, 2, \ldots$ are the probabilities of an ordinary Poisson random variable.

Now consider the random variable X defined by

$$X = Z \cdot Y$$

where $Z \sim \text{Bernoulli}(1-\pi)$, $Y \sim \mathcal{PN}(\lambda)$, and Z, Y are independent random variables.

(a) Prove that N and X are equivalent random variables by showing that the probability mass function of X is exactly the same as that of N, as given above.

(b) Use this result to show that the mean of N is

$$\mathbb{E}[N] = \mu = (1-\pi)\lambda$$

and its variance is

$$\text{Var}(N) = \mu + \frac{\pi}{1-\pi}\mu^2.$$

(c) A variation of the method of moments is to match $\mathbb{E}[N]$ with sample mean and $\text{Var}(N)$ with sample variance. Given observed data $x_1, x_2, \ldots, x_n$, use this suggested method of moments to show that resulting moment estimators of π and λ respectively can be expressed as

$$\widetilde{\pi}_{\text{ME}} = \frac{s^2 - \overline{x}}{\overline{x}^2 + (s^2 - \overline{x})}$$

and

$$\widetilde{\lambda}_{\text{ME}} = \overline{x} + \left(\frac{s^2}{\overline{x}} - 1\right),$$

where

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

are the respective sample mean and sample variance.

(d) Explain, in words, why zero-inflated Poisson is more realistic and applicable to insurance losses than ordinary Poisson.

— end —

APPENDIX

A random variable X is said to have a two-parameter Weibull distribution if its density has the form

$$f(x) = \frac{1}{x}\tau(x/\theta)^{\tau}e^{-(x/\theta)^{\tau}}, \quad \text{for } x > 0.$$

This distribution satisfies the following:

$$\mathrm{E}[X^k] = \theta^k\Gamma(1 + k/\tau), \quad \text{for any } k > -\tau.$$

and

$$\mathrm{E}[(X \wedge x)^k] = \theta^k\Gamma(1 + k/\tau)\Gamma[1 + k/\tau; (x/\theta)^{\tau}] + x^k e^{-(x/\theta)^{\tau}}, \quad \text{for any } k > -\tau.$$

A discrete random variable N is said to belong to the $(a, b, 0)$ class of distributions if it satisfies the relation

$$\Pr(N = k) = p_k = \left(a + \frac{b}{k}\right) \cdot p_{k-1}, \quad \text{for } k = 1, 2, \ldots,$$

for some constants a and b. Alternatively, this relation can be expressed as a linear function given by

$$k \cdot \frac{p_k}{p_{k-1}} = b + ak, \quad \text{for } k = 1, 2, \ldots.$$

The initial value p_0 is determined so that $\sum_{k=0}^{\infty} p_k = 1$.

The Poisson distribution $\mathcal{PN}(\lambda)$ belongs to the $(a, b, 0)$ class of distributions with $a = 0$ and $b = \lambda$. Its mean and variance are:

$$\mathbb{E}[X] = \lambda \quad \text{and} \quad \mathrm{Var}(X) = \lambda.$$