Probability Prelim Exam for Actuarial Students
January 2025

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for mathematically incorrect statements.

In what follows $(\Omega, \mathcal{F}, P)$ is a probability space. Events are elements in $\mathcal{F}$.

1. (10 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $E[X_i] = 0$ and $\text{Var}(X_i) = 1$ for all i. Show that
$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{X_k \geq k\}\right) = 0.$$

2. (10 points) Let $Z, Z_1, Z_2, \ldots$ be random variables. Suppose that for each $\epsilon > 0$, we have
$$\sum_{n=1}^{\infty} P(|Z_n - Z| \geq \epsilon) < \infty.$$
Show that $Z_n \to Z$ almost surely.

3. (10 points) Let X be a random variable with $E[X] = 0$ and $E[X^2] = 1$. Show that
$$P(X \geq 1) \leq \frac{1}{2}.$$

4. (10 points) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables on a probability space $(\Omega, \mathscr{F}, P)$ with
$$P(X_n = 1) = P(X_n = 0) = \frac{1}{4}, \quad P(X_n = -1) = \frac{1}{2}.$$
Let a be a positive integer, $S_0 = a$, and
$$S_n = a + \sum_{i=1}^{n} X_i, \quad n \geq 1.$$
Let $\tau_0 = \inf\{n \geq 0 : S_n = 0\}$. Calculate $P(\tau_0 < \infty)$.

5. (10 points) Let X be a random variable on a probability space $(\Omega, \mathscr{F}, P)$. Suppose that X takes only nonnegative integer values. Show that
$$E[X] = \sum_{k=0}^{\infty} P\{X > k\}.$$

6. (10 points) Let $\{M_n\}_{n=0,1,2,\ldots}$ be a symmetric random walk, i.e.,

$$M_0 = 0, \quad M_n = Z_1 + Z_2 + \cdots + Z_n \quad n \geq 1,$$

where

$$P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2}, \quad n \geq 1.$$

Define a new process $\{I_n\}_{n\geq 0}$ by letting $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} M_j \big(M_{j+1} - M_j\big), \quad n = 1, 2, \ldots.$$

First show that $I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$ and use this identity to show that $\{I_n\}_{n\geq 0}$ is a martingale.

7. (10 points) Let $\{B_t : t \geq 0\}$ be a standard Brownian motion. For some positive integer n, let Z_n be defined as

$$Z_n = \frac{B_1 + B_2 + \cdots + B_n}{n}.$$

Compute the distribution of Z_n.