Real Analysis Preliminary Exam
January 2025

Instructions and Notation

- Justify all your steps, clearly identifying all results you are using. Do not invoke a result that is essentially equivalent to what you are asked to prove.
- The Lebesgue σ-algebra on $\mathbb{R}$ and the Lebesgue measure on $\mathbb{R}$ are denoted by $\mathcal{L}$ and m, respectively.

Problems

1. (10 points) Let $(\Omega, \mathcal{F})$ be a measurable space and let f be a real-valued measurable function. For $M > 0$, let $A_M = \{x \in \Omega : |f(x)| \leq M\}$. Prove that there exists a sequence of simple functions $(\phi_n : n \in \mathbb{N})$ with the property that for every $M > 0$ there exists $N \in \mathbb{N}$ so that $\sup_{x \in A_M} |\phi_n(x) - f(x)| < \frac{1}{M}$ for all $n \geq N$.

2. (10 points) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $(f_n : n \in \mathbb{N})$ and f measurable functions satisfying $\lim_{n\to\infty} f_n = f$ in μ-measure and
$$\limsup_{n\to\infty} \int |f_n| d\mu \leq \int |f| d\mu = 1.$$
Prove
$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

3. (10 points) Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and let $f : \Omega \to [0, \infty]$ be an nonnegative integrable function. For $x \geq 0$, let $h(x) = \mu(\{f = x\})$. Prove that h is continuous and equal to zero, possibly except on a countable (this includes empty and finite) set.

4. (10 points) Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Suppose that $f : \Omega \to [0, \infty)$ is a measurable function, and that $\varphi : [0, \infty) \to \mathbb{R}$ an absolutely continuous function with $\varphi' \geq 0$, m-a.e. Prove

   (a). The set
   $$B = \{(\omega, s) \in \Omega \times \mathbb{R} : f(\omega) \leq s\}$$
   is measurable in the product σ-algebra $\mathcal{F} \times \mathcal{L}$.

   (b).
   $$\int \varphi(f) d\mu = \varphi(0)\mu(\Omega) + \int_{[0,\infty)} \mu(f \geq s)\varphi'(s) dm(s).$$

5. (10 points) Let f be a Lebesgue integrable function on $\mathbb{R}$ and let $h(y) = \int |f(x-y) - f(x)| dm(x)$. Prove
$$\lim_{y\to\infty} h(y) = 2 \int |f| dm.$$

6. (10 points) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose that $n \geq 2$ and $p_1, p_2, \ldots p_n$ are positive reals satisfying $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$ and $f_j \in L^{p_j}(\mu)$ for $j = 1, \ldots, n$. Prove that $\prod_{j=1}^n f_j \in L^1(\mu)$ and
$$\| \prod_{j=1}^n f_j \|_1 \leq \prod_{j=1}^n \|f_j\|_{p_j}.$$

7. (10 points) Let $g : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function, and $\mathcal{G}$ be the σ-algebra generated by g, that is the intersection of all σ-algebras containing the sets $L_y = \{x \in \mathbb{R} : g(x) \leq y\}$ where y ranges over $\mathbb{R}$. Show that for any Lebesgue integrable f there exists a $\mathcal{G}$-measurable function H_f such that
$$\int_A f dm = \int_A H_f dm, \quad \text{for all } A \in \mathcal{G}.$$