Complex Analysis Preliminary Exam, August 2025

Instructions and notation:

(i) Give full justifications for all solutions in the exam booklet. Four completely correct solutions will guarantee a Ph.D. pass.

(ii) Clearly state which theorems you are using and verify their assumptions.

(iii) For $p \in \mathbb{C}$ and $R > 0$, $B(p, R) = \{z \in \mathbb{C} : |z - p| < R\}$ and $\bar{B}(p, R) = \{z \in \mathbb{C} : |z - p| \le R\}$.

1. Let $S = \{x + iy \in \mathbb{C} : -1 < x < |\cos x|\}$. If $f : S \to S$ is holomorphic and $f(0) = 0$ show that $|f'(0)| \le 1$.
2. Show that if f is entire and satisfies $|f'(z)| \le |z|, z \in \mathbb{C}$, then
$$f(z) = a + bz^2 \text{ for some } a \in \mathbb{C} \text{ and } |b| \le 1/2.$$
3. Let f be a non-constant entire function. Show that $\lim_{|z|\to\infty} |f(z)| = +\infty$ if and only if f is a non-constant polynomial. Recall that $\lim_{|z|\to\infty} |f(z)| = +\infty$ if and only if for every $M > 0$ there exists $R > 0$ such that $|f(z)| > M$ for all $z \in \mathbb{C} \backslash B(0, R)$.
4. Evaluate
$$\int_\gamma \frac{z^{24}}{100z^{25} - 23z^{20} + 7z^{15} - 5} dz,$$
where $\gamma(t) = e^{it}, t \in [0, 2\pi]$.
5. Find all holomorphic functions $f : B(0, 1) \to \mathbb{C}$ such that
$$f''(1/n) + f(1/n) = 0, \qquad \text{for } n = 2, 3, 4, \ldots.$$
6. Show that if f is entire and satisfies:
   (a) $\operatorname{Im} z > 0 \implies \operatorname{Im} f(z) > 0$,
   (b) $\operatorname{Im} z = 0 \implies \operatorname{Im} f(z) = 0$,
   (c) $\operatorname{Im} z < 0 \implies \operatorname{Im} f(z) < 0$,

   then there exists $w \in \mathbb{C}$ such that $f(w) = 0$.
7. Find all harmonic functions $u : \mathbb{C} \to \mathbb{R}$ such that $\frac{\partial u}{\partial x} \ge 2025$.