GEOMETRY & TOPOLOGY PRELIMINARY EXAM, AUGUST 2025

Problem 1. (1) Let X and Y be two topological spaces. Prove that a function $f : X \to Y$ is continuous if and only if for any subset $A \subset Y$,
$$\overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$$
where $\overline{A}$ is the closure of A.

(2) Let A and B be subsets of a topological space X so that $A \cup B$ and $A \cap B$ are connected. Prove that if A and B are closed, then both A and B are connected.

Problem 2. Let X and Y be topological spaces, and let $\pi_X : X \times Y \to X$ be the projection map.

(1) Give an example showing that, in general, π_X need not be a closed map.
(2) Prove that if Y is compact, then π_X is a closed map.

Problem 3. Let $\mathbb{P}^n$ be the real projective space, defined as the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation:
$$(x_0, \cdots, x_n) \sim (y_0, \ldots, y_n) \quad \text{if and only if} \quad (y_0, \cdots, y_n) = \lambda(x_0, \cdots, x_n)$$
for some nonzero real number λ. For a nonzero vector $(x_0, \ldots, x_n)$, we denote its equivalence class in $\mathbb{P}^n$ as
$$[x_0 : \cdots : x_n] = \{(\lambda x_0, \ldots, \lambda x_n) : \lambda \in \mathbb{R} \setminus \{0\}\}.$$

(1) Show that the sets
$$U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i \neq 0\}, \quad i = 0, \ldots, n$$
form an open cover of $\mathbb{P}^n$.
(2) Prove that each U_i is homeomorphic to $\mathbb{R}^n$.

Problem 4. Let $f : X \to Y$ be a continuous bijection, where X is compact.

(1) If Y is Hausdorff, show that f is a homeomorphism.
(2) If the assumption that Y is Hausdorff is removed in (1), is f still a homeomorphism? Prove it or give a counter-example.

Problem 5. Let $X \subseteq \mathbb{R}^3$ be the union of the unit 2-sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with the line segment $\{(0, 0, z) : -1 \leq z \leq 1\}$. Compute the fundamental group of X based at the north pole $(0, 0, 1)$, giving explicit generator(s).

Problem 6. Let $n \geq 1$ and $\mathbb{S}^n$ be the n-dimensional sphere. Suppose $f : \mathbb{S}^n \to \mathbb{S}^n$ and $g : \mathbb{S}^n \to \mathbb{S}^n$ are continuous maps such that $f(x) \neq g(x)$ for all $x \in \mathbb{S}^n$. Prove that f is homotopic to the antipodal map of g; that is, $f \sim -g$.