Real Analysis Prelim (Math 5111)

University of Connecticut

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Instructions: Do as many of the following problems as you can—and always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill in the gap. You may use any standard theorem from the real analysis course, identifying it by name or stating it in full. You may not use a cannon to kill a mosquito. A completely correct solution is worth more than two partial solutions. Problems will be graded as follows.

- 10 points: solution is completely correct or essentially correct modulo harmless errors
- 8 points: solution has a correct outline, but has a few minor to moderate errors or omits justifications
- 4 points: solution is incomplete, but does not make any significant errors and has correct details for substantial parts of the argument
- 1 point: solution is incomplete and makes a serious error or false claim, but has correct details for some parts of the argument
- 0 points: solution is essentially incorrect or makes several serious errors or false claims

A total score of 36 points or more will guarantee a Ph.D. pass

Notation and Conventions:

- Below m^n denotes the Lebesgue outer measure on $\mathbb{R}^n$ and $\mathcal{L}^n$ denotes the set of m^n measurable sets. If the dimension of the ambient space is clear, you may write m instead of m^n.
- $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$

Problems:

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Find a general formula to evaluate $\mu(A \cup B \cup C)$ for all $A, B, C \in \mathcal{M}$ in terms of the μ measure of the sets A, B, C, and sets that can be written using finite intersections of A, B, or C (e.g. $A \cap C$ or $A \cap B \cap C$). Prove that your formula is correct.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that if $A_1, A_2, \cdots \in \mathcal{M}$ and $\prod_{i=1}^{\infty} e^{\mu(A_i)} < \infty$, then $S = \{x \in X : x \in A_i \text{ for infinitely many } i\} \in \mathcal{M}$ and $\mu(S) = 0$.

3. Suppose that $f : \mathbb{R}^1 \to \mathbb{R}^2$ satisfies the inequality $|f(x) - f(y)| \leq |x - y|^{1/2}$ for every $x, y \in \mathbb{R}^1$. Prove that if $A \in \mathcal{L}^1$, then $f(A) \in \mathcal{L}^2$ and $m^2(f(A)) \leq C m^1(A)$ for some numerical constant $C \in (0, \infty)$ that is independent of A.

4. Let $U(x,r) = \{y \in \mathbb{R}^3 : |x-y| < r\}$ and $B(x,r) = \{y \in \mathbb{R}^3 : |x-y| \leq r\}$ denote the open and closed balls with center $x \in \mathbb{R}^3$ and radius $r > 0$. Prove that $m^3(U(x,r)) = m^3(B(x,r))$ for every $x \in \mathbb{R}^3$ and every $r > 0$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 < p < \infty$. Prove that if f and g are measurable functions that take values in $[0,\infty]$ and $f, g \in L^p(\mu)$, then

$$\int f^{p-1} g \, d\mu < \infty \quad \text{and} \quad \int (f + \varepsilon g)^p \, d\mu = \underbrace{\int f^p \, d\mu}_{I} + \underbrace{p\varepsilon \int f^{p-1} g \, d\mu}_{II} + \underbrace{o(\varepsilon) \cdot \varepsilon}_{III}, \tag{E5}$$

where $\varepsilon > 0$ and the error term $o(\varepsilon) \to 0$ as $\varepsilon \to 0+$.

6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Prove that $L^{25}(\mu) \subseteq L^{20}(\mu)$ if and only if $\mu(X) < \infty$. For this problem, the convention is that functions in $L^p(\mu)$ are measurable and take values in $[-\infty, \infty]$.

7. Let μ and ν be probability measures on a measurable space $(X, \mathcal{M})$ such that $\mu \ll \nu$. Let $\mu \times \mu$ and $\nu \times \nu$ denote product measures on $\mathcal{M} \otimes \mathcal{M}$. Prove that $\mu \times \mu$ and $\nu \times \nu$ are probability measures and $\mu \times \mu \ll \nu \times \nu$. Then establish the following relationship between the Radon-Nikodym derivatives:

$$\frac{d(\mu \times \mu)}{d(\nu \times \nu)}(x,y) = \frac{d\mu}{d\nu}(x) \cdot \frac{d\mu}{d\nu}(y) \quad \text{for } \nu\text{-a.e. } x, y \in X. \tag{E7}$$

— End of Problems —