Abstract Algebra Prelim Aug. 2025

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) Let G be a finite cyclic group with order n.
   (a) (**5 pts**) Prove every subgroup of G is cyclic.
   (b) (**5 pts**) If $d \mid n$, then prove (i) the subgroup $\langle g^{n/d} \rangle$ has order d and (ii) $\langle g^{n/d} \rangle$ is the only subgroup of order d.

2. (**10 pts**) Let G be a finite group. For a prime p, let $|G| = p^k m$ where $k \geq 0$ and $p \nmid m$, and let n_p be the number of Sylow p-subgroups of G.

   Prove the *third* Sylow theorem, which has two statements:

   (i) $n_p \equiv 1 \bmod p$,

   (ii) if P is a Sylow p-subgroup of G, then $n_p = [G : \mathrm{N}_G(P)]$ and $n_p \mid m$, where $\mathrm{N}_G(P)$ is the normalizer of P in G.

   Your solution may use without proof the first two Sylow theorems and properties of group actions. Indicate clearly which group actions you rely on.

3. (**10 pts**)
   (a) (**4 pts**) In a commutative ring A, define prime ideals and maximal ideals.
   (b) (**6 pts**) In $\mathbf{Z}[x]$, show the ideal $(3, x^2 - 2)$ is maximal while the ideal $(7, x^2 - 2)$ is neither maximal nor prime.

4. (**10 pts**)

   Every $n \times n$ real matrix A defines a linear map $\mathbf{R}^n \to \mathbf{R}^n$ in the usual way, by $\mathbf{v} \mapsto A\mathbf{v}$. The transpose of A is denoted $A^\top$.

   (a) (**4 pts**) Show $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^\top \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$.
   (b) (**6 pts**) Use the identity in (a) to show $\ker(A^\top) = (\mathrm{im}(A))^\perp$: in $\mathbf{R}^n$, the kernel of $A^\top$ is the orthogonal complement of the image of A.

5. (**10 pts**) Let V be a finite-dimensional vector space over a field K and W be a subspace of V.
   (a) (**5 pts**) Each φ in the dual space V^* can be restricted to W^*. Let $R_{V,W} : V^* \to W^*$ be this restriction: $R_{V,W}(\varphi) = \varphi|_W$. Prove $R_{V,W}$ is linear and surjective.
   (b) (**5 pts**) The kernel of $R_{V,W}$ is $\{\varphi \in V^* : \varphi = 0 \text{ on } W\}$. Each $\varphi \in \ker R_{V,W}$ induces a linear map $\overline{\varphi} : V/W \to K$ where $\overline{\varphi}(v \bmod W) = \varphi(v)$, so $\overline{\varphi} \in (V/W)^*$. Prove the mapping $\ker R_{V,W} \to (V/W)^*$ where $\varphi \mapsto \overline{\varphi}$ is a vector space isomorphism. (Hint: By part (a), $V^*/\ker R_{V,W} \cong W^*$, so describe $\dim(\ker R_{V,W})$ in terms of $\dim V$ and $\dim W$.)

6. (**10 pts**) Give examples as requested, with justification.
   (a) (**2.5 pts**) A description of S_n, for each $n \geq 3$, as a semidirect product of two nontrivial subgroups.
   (b) (**2.5 pts**) A polynomial $f(x) \in \mathbf{Q}[x]$ such that $\mathbf{Q}[x]/(f(x)) \cong \mathbf{Q} \times \mathbf{Q}$ as rings.
   (c) (**2.5 pts**) A quadratic ring $\mathbf{Z}[\sqrt{d}]$, meaning d in $\mathbf{Z}$ is not a perfect square, whose only units are ± 1.
   (d) (**2.5 pts**) An inner product on $\mathbf{R}^n$ besides the standard inner product.