Applied Math Prelim August 2025

1. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a linear bounded operator. Suppose $\{u_n\}_{n=1}^{\infty}$ is any weakly converging sequence.

   (a) (4 pts) If $u_n \rightharpoonup u_0$, show that $Tu_n \rightharpoonup Tu_0$.

   (b) (5 pts) Show that $\{u_n\}$ is bounded.

   (c) (4 pts) Explain what weak (sequentially) compactness of a bounded sequence in $\mathcal{H}$ is.

   (d) (7 pts) Show that T is a compact operator if and only if T maps any weakly converging sequence to a converging sequence. The parts above may help.

2. Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence.

   (a) (9 pts) Suppose $T_n : \mathcal{H} \to \mathcal{H}$ is a linear bounded compact operator for all $n = 1, 2, \ldots$ and $T_n \to B$ in operator norm for some linear bounded operator $B : X \to X$. Show that B is a compact operator.

   (b) (6 pts) Given $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a bounded sequence. Let $Bx := \sum_j \lambda_j \langle x, e_j \rangle \, e_j$. Using part (a) or otherwise, show that B is a linear bounded compact operator iff $\lambda_n \to 0$.

   (c) (5 pts) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in the Hilbert space X. Suppose $B : X \to X$ is a linear bounded operator with $\sum_{j=1}^{\infty} \|Be_j\|^2 < \infty$. (This is known as the Hilbert-Schmidt operator). For any $w \in X$, define $T_n : X \to X$ such that $T_n w = \sum_{j=1}^{n} \langle w, e_j \rangle Be_j$. Show that T_n is a linear bounded compact operator and $T_n \to B$ in operator norm.

3. For any $f \in L^1_{loc}(\mathbb{R}^n)$ we let $\tilde{f} \in \mathcal{D}'(\mathbb{R}^n)$ be the distribution such that $\tilde{f}(\phi) = \int_{\mathbb{R}^n} f(z)\varphi(z)\, dz$ for any test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

   (a) (8 pts) Let $\{f_j\}_{j=1}^{\infty} \subset L^1(\mathbb{R}^n)$ be non-negative with $\int_{\mathbb{R}^n} f_j(z)\, dz = 1$ and $\lim_{j\to\infty} \int_{|x|\geq r} f_j(x)\, dx = 0$ for any $r > 0$. Show that $\tilde{f}_j \to \delta$, where δ is the Dirac distribution (delta function).

   (b) (4 pts) Let $f \in L^1(\mathbb{R}^n)$ be non-negative with $\int_{\mathbb{R}^n} f(z)\, dz = 1$. Let $f_j(z) := j^n f(jz)$. Show that $\{f_j\}$ satisfies the assumptions in part (a).

(c) (2 pts) For $n = 1$ and

$$f(x) = \begin{cases} 1, & \text{if } |x| < 1/2\,, \\ 0, & \text{otherwise.} \end{cases}$$

What is $\{f_j\}$ in part (b)?

(d) (6 pts) Let $n = 1$. Find a distribution T such that $\partial^2 T - T = \delta$. Is this distribution unique? Explain.

4. (a) (8 pts) Find the Green's function $G(x, y)$ for the operator A where

$$Au := u''$$

with $u(0) = u(\pi) = 0$. Show that $G(x, y) = G(y, x)$ for all $x, y \in [0, \pi]$.

(b) (5 pts) Define $K : L^2(0, \pi) \to L^2(0, \pi)$ such that for any $f \in L^2(0, \pi)$,

$$(Kf)(x) := \int_0^\pi G(x, y) f(y)\, dy\ .$$

Show that K is a linear bounded self-adjoint compact operator. (For compactness you can use the fact if $\int_0^\pi \int_0^\pi |G(x, y)|^2\, dxdy < \infty$, then K is a Hilbert-Schmidt operator in question (2c) and therefore compact).

(c) (7 pts) Show that the range of $I + K$ is closed, where I is the identity operator and K is any linear bounded compact operator on any Hilbert space.

5. Let $F : D \to Y$ be a mapping from an open set D in a Banach space X to another Banach space Y.

(a) (5 pts) State the definition of Fréchet derivative of F.

(b) (10 pts) Let $D = X = \{u \in C^2[0, 1] : u(0) = u(1) = 0\}$ equipped with C^2 norm. For any $u \in X$, define $F : X \to C[0, 1]$ such that for any $u \in X$

$$F(u) := u'' + u - u^3\ .$$

Show that F is Fréchet differentiable at any $u \in C[0, 1]$ and find its derivative.

(c) (5 pts) Given a C^1 function $g : \mathbb{R} \to \mathbb{R}$ (e.g. $g(t) = t - t^3$ in part (b)). Calculate the Fréchet derivative for $F(u) := u'' + g(u)$.